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Stability Analysis of Switched Dynamical Systems with State-Space Dilation and Contraction

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This paper considers switched dynamical systems with state-space dilation and contraction formed by concatenating the states of a set of local dynamical systems or semiflows on state spaces with different dimensions at specified time instants. These systems arise naturally in many aerospace applications, such as multibody dynamic systems involving changes in the degrees of freedom and systems composed of multiple spacecraft with docking and undocking capabilities flying in formation. The notions of stability of invariant sets in the sense of Lyapunov for usual dynamical systems are extended to this class of switched dynamical systems. Also, the notion of finite-time stability with specified bounds is introduced. Sufficient stability conditions are established for a special class of switched dynamical systems with state-space dilation and contraction. Their application to spacecraft formation stability analysis is discussed.

I. Introduction

SWITCHED dynamical systems with state-space dilation and contraction arise naturally in many aerospace applications. An example of such systems is a spacecraft formation in which each spacecraft has the capability of docking with one or more spacecraft to form an aggregated spacecraft [1]. The latter may also disassemble into a number of smaller spacecraft via undocking. Here, the dimension of the state space of the spacecraft formation [2] at any time depends on the total number of spacecraft in the formation at that time, and the number of degrees of freedom of each aggregated or disassembled spacecraft. In the framework of classical mechanics, its state space can be taken as the Cartesian product of the configuration and generalized momentum spaces of the spacecraft.

Although there are numerous works on the stability and control of switched and hybrid systems, almost all of them deal with systems with a single state space [3–10]. Recently, a few basic issues in the modelling and analysis of switched or hybrid dynamical systems with state-space dilation and contraction such as their mathematical descriptions, controllability, and observability [11] were considered. In this paper, we focus our attention on the stability analysis of switched dynamical systems with state-space dilation and contraction.

II. Preliminaries

We begin with a mathematical description of switched dynamical systems with state-space dilation and contraction.

Let $I_k = [t_{k-1}, t_k]$, $k = 1, \ldots, K$ be a specified finite sequence of consecutive disjoint time intervals, and $\bar{I}_k = [t_{k-1}, t_k]$, where t_K may be finite or infinite. For any time interval I_k, the system (also referred to hereafter as the kth elemental system) is described by an ordinary differential equation:

$$d\mathbf{x}^{(k)}/dt = \mathbf{f}^{(k)}(t, \mathbf{x}^{(k)}) \tag{1}$$

where the system state $\mathbf{x}^{(k)}(t)$ at any time $t \in I_k$ corresponds to a point in the state-space Σ_k, a subset of a n_k-dimensional normed linear space over the scalar field F (same for all Σ_k) with norm $\|\cdot\|^{(k)}$; $\mathbf{f}^{(k)}$ is a specified function of its arguments such that $\mathbf{f}^{(k)}(t, \mathbf{0}) = \mathbf{0} \in \Sigma_k$ for all $t \in \bar{I}_k$. At $t = t_{k-1}$, the states of the kth and $(k-1)$th elemental systems are related by

$$\mathbf{x}^{(k)}(t_{k-1}) = \mathbf{S}^{(k,k-1)}(\mathbf{x}^{(k-1)}(t_{k-1})) \tag{2}$$

where $\mathbf{S}^{(k,k-1)}$ is the state concatenation operator (a continuous mapping on $\Sigma_{k-1} \to \Sigma_k$). The continuity of $\mathbf{S}^{(k,k-1)}$ is defined with respect to the norms of Σ_k and Σ_{k-1}. Note that the dimension of the range of $\mathbf{S}^{(k,k-1)}$ is generally less than or equal to the dimension of Σ_k.

Remark 2.1: A more general description of the foregoing system is to define a family $\mathcal{E}$ of elemental systems with different state spaces. Each elemental system is uniquely labeled by a positive integer from the index set $\mathcal{I}$. We define a piecewise constant switching function $s = s(t)$ on the time interval $T_K = \bigcup_{k=1}^{K} I_k$ such that on each subinterval I_k, s takes on a constant value in $\mathcal{I}$. For each ordered pair of elemental systems in $\mathcal{E}$ or the corresponding index pair $(i, j) \in \mathcal{I} \times \mathcal{I}$, a unique concatenation operator $\mathbf{S}^{(i,j)}$ is defined. Thus, for a given switching function $s = s(t)$, a unique sequence pair of elemental systems and concatenation operators in the form of Eqs. (1) and (2) is defined.

Remark 2.2: In the case in which the state spaces $\Sigma_k \subseteq \Sigma_{k^*}$, $k = 1, \ldots, K$, where k^* is the $k \in \{1, \ldots, K\}$ such that $n_{k^*} = \max\{n_k, k = 1, \ldots, K\}$, we can regard the concatenated system as a single switched system with state space Σ_{k^*}. However, from the practical standpoint, it may not be desirable to deal with the representation of the system with the highest dimension at all times. Of course, such a representation does not exist if $\Sigma_k \cap \Sigma_{k^*}$ is empty for some or all $k \neq k^*$.

Remark 2.3: In more complex situations, the switching time instant t_k depends on the system state, as in hybrid systems [6]. This case will not be considered here.

We assume that Eq. (1) generates a local semidynamical system or local semiflow $\mathcal{F}_k = \{\Sigma_k, \mathcal{G}_k\}$ on I_k, where $\mathcal{G}_k = \{\mathbf{\Phi}^{(k)}_{t,t'}\colon t \geq t'; t, t' \in$

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$\bar{I}_k\}$ is a two-parameter family of evolution or state transition operators $\mathbf{\Phi}^{(k)}_{t,t'}$ on the state space Σ_k into Σ_k. The evolution operators satisfy 1) $\mathbf{\Phi}^{(k)}_{t,t} = \mathbf{I}^{(k)}$ (the identity operator on Σ_k), and 2) $\mathbf{\Phi}^{(k)}_{t,\tau} \circ \mathbf{\Phi}^{(k)}_{\tau,t'} = \mathbf{\Phi}^{(k)}_{t,t'}$ for any $t, \tau, t' \in \bar{I}_k$, and $t \geq \tau \geq t'$. Moreover, $\mathbf{\Phi}^{(k)}_{t,t_{k-1}}: \Sigma_k \to \Sigma_k$ is uniformly continuous on $\bar{I}_k$. The term "local" refers to the restriction of the time interval of the definition of $\mathcal{F}_k$ to $\bar{I}_k$. The foregoing assumption implies the existence and uniqueness of the solutions of Eq. (1) on I_k, and the continuous dependence of the solutions on the initial state at t_{k-1}. This assumption can be satisfied if $\mathbf{f}^{(k)}$ is sufficiently smooth or satisfies a uniform Lipschitz condition with respect to $\mathbf{x}^{(k)}$ on I_k. For a given finite set of local semiflows $\mathcal{F}_k = \{\Sigma_k, \mathcal{G}_k\}$, $k = 1, \ldots, K$, the set $\phi_k^+(\mathbf{x}^{(k)}(t_{k-1}))= \{\mathbf{\Phi}^{(k)}_{t,t_{k-1}}(\mathbf{x}^{(k)}(t_{k-1})) \in \Sigma_k\colon t \in \bar{I}_k\}$ will be referred to as an orbital segment of $\mathcal{F}_k$ emanating from $\mathbf{x}^{(k)}(t_{k-1})$.

Definition 2.1: The local semiflow pair $\{\mathcal{F}_k, \mathcal{F}_{k+1}\}$ is said to be *dilative* (respectively, *contractive*), if the dimensions of Σ_k and Σ_{k+1} satisfy $n_k < n_{k+1}$ (respectively, $n_k > n_{k+1}$).

We construct a switched semiflow or switched semidynamical system over $T_K = \bigcup_{k=1}^K I_k$ by concatenating a string of local semiflows $\mathcal{F}_k$, $k = 1, \ldots, K$, and defining a family of state transition operators by

$$\mathbf{\Phi}_{t,t'} = \mathbf{\Phi}^{(k)}_{t,t_{k-1}} \circ \left(\mathbf{S}^{(k-1,k-2)} \circ \mathbf{\Phi}^{(k-1)}_{t_{k-1},t_{k-2}}\right) \circ \left(\mathbf{S}^{(k-2,k-3)} \circ \mathbf{\Phi}^{(k-2)}_{t_{k-2},t_{k-3}}\right) \circ \cdots \circ \left(\mathbf{S}^{(k'+1,k')} \circ \mathbf{\Phi}^{(k')}_{t_{k'},t'}\right) \tag{3}$$

for $t \geq t'$; $t \in I_k$, $t' \in I_{k'}$; $I_k \cap I_{k'} = \phi$ for $k \neq k'$; and $\mathbf{\Phi}_{t,t'} = \mathbf{\Phi}^{(k')}_{t,t'}$ for $t \geq t'$; $t, t' \in I_{k'}$. Then, $\mathcal{F} \stackrel{\text{def}}{=} \{\Sigma_1, \mathcal{G}\}$ is a semiflow over T_K, where $\mathcal{G} = \{\mathbf{\Phi}_{t,t'}\colon t \geq t'; t, t' \in T_K\}$.

Remark 2.4: Definition 2.1 and the foregoing construction of a switched semiflow via concatenation of local semiflows remain valid when the local semiflows are replaced by local flows, that is, to each element $\mathbf{\Phi}^{(k)}_{t,t'}$, $t, t' \in \bar{I}_k$, in $\mathcal{G}_k$, an inverse element is defined by $(\mathbf{\Phi}^{(k)}_{t,t'})^{-1} = \mathbf{\Phi}^{(k)}_{t',t}$. However, because the state concatenation operators $\mathbf{S}^{(k+1,k)}$ are not invertible, the concatenation of a sequence of local flows by means of Eq. (3) only results in a semiflow.

Remark 2.5: In physical situations such as assembly or disassembly of space structures composed of dynamic structural elements, the switched semiflow $\mathcal{F}$ may be formed by concatenating a string of successively contracting or dilating local semiflow pairs, respectively. In this case, we have semiflows with telescopically contracting/dilating state spaces.

III. Stability

Now, we extend the notions of limit sets and invariant sets in usual semiflows to those in switched semiflows with state-space dilation and contraction. In what follows, we first consider a switched semiflow $\mathcal{F} = \{\Sigma_1, \mathcal{G}\}$ with sequential state-space dilation and contraction formed by concatenating a finite number of local semiflows $\mathcal{F}_k$ over $\bar{I}_k$, $k = 1, \ldots, K$.

To extend the notion of a positively invariant set to switched semiflows with state-space dilation and contraction, we first define a local positively invariant set as follows:

Definition 3.1: A set $\Gamma_k \subseteq \Sigma_k$ is said to be *positively invariant* with respect to the local semiflow $\mathcal{F}_k$ if for any $\mathbf{x}^{(k)}(t_{k-1}) \in \Gamma_k$, its orbital segment $\phi_k^+(\mathbf{x}^{(k)}(t_{k-1}))$ emanating from $\mathbf{x}^{(k)}(t_{k-1})$ lies in Γ_k.

Definition 3.2: A set $\Gamma_1 \subseteq \Sigma_1$ is said to be *positively invariant* with respect to the switched semiflow $\mathcal{F} = \{\Sigma_1, \mathcal{G}\}$ if the set $\Gamma_{k+1} \stackrel{\text{def}}{=} \mathbf{S}^{(k+1,k)}(\Gamma_k)$ is positively invariant with respect to $\mathcal{F}_{k+1}$ for all $k = 1, \ldots, K-1$.

Definition 3.3: A point $\mathbf{x}^{(1)}_{\text{eq}} \in \Sigma_1$ is an *equilibrium state* of the switched semiflow $\mathcal{F} = \{\Sigma_1, \mathcal{G}\}$ if for each $k \in \{1, \ldots, K\}$, $\mathbf{\Phi}^{(k)}_{t,t_{k-1}}(\mathbf{x}^{(k)}_{\text{eq}}) = \mathbf{x}^{(k)}_{\text{eq}}$ for all $t \in I_k$, and $\mathbf{x}^{(k+1)}_{\text{eq}} \stackrel{\text{def}}{=} \mathbf{S}^{(k+1,k)}(\mathbf{x}^{(k)}_{\text{eq}})$ for $k = 1, \ldots, K-1$.

Clearly, if $\mathbf{x}^{(1)}_{\text{eq}} \in \Sigma_1$ is an equilibrium state of the switched semiflow $\mathcal{F}$ generated by differential equations in Eq. (1), then for each $k \in \{1, \ldots, K\}$, $\mathbf{f}^{(k)}(t, \mathbf{x}^{(k)}_{\text{eq}}) = \mathbf{0}$ for all $t \in I_k$.

A. Lyapunov Stability

Now, we give the definition for Lyapunov stability of positively invariant sets for switched semiflows with state-space dilation and contraction.

Definition 3.4: A positively invariant set $\Gamma_1 \subset \Sigma$ of a switched semiflow $\mathcal{F} = \{\Sigma_1, \mathcal{G}\}$ is said to be *stable* if given any real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$\rho_1(\mathbf{x}^{(1)}(t_0), \Gamma_1) < \delta \Rightarrow \sup_{t \in I_k} \rho_k\left(\mathbf{\Phi}^{(k)}_{t,t_{k-1}}(\mathbf{x}^{(k)}(t_{k-1})), \Gamma_k\right) < \epsilon \tag{4}$$

for all $k = 1, \ldots, K$, where $\rho_k(\mathbf{x}, \Gamma_k) \stackrel{\text{def}}{=} \inf\{\|\mathbf{x} - \mathbf{x}'\|^{(k)}, \mathbf{x}' \in \Gamma_k\}$ and $\Gamma_{k+1} = \mathbf{S}^{(k+1,k)}(\Gamma_k)$, $k = 1, \ldots, K-1$. If, in addition,

$$\rho_K\left(\mathbf{\Phi}^{(K)}_{t,t_{K-1}}(\mathbf{x}^{(K)}(t_{K-1})), \Gamma_K\right) \to 0 \tag{5}$$

as $t \to t_K$ (t_K may be finite or infinite), then the positively invariant set Γ_1 is said to be *asymptotically stable*.

Note that for $t_K < \infty$, condition (5) implies convergence in finite time. This form of finite-time asymptotic stability is attainable for a certain class of switched semidynamical systems with nonlinear feedback controls.

Now, for any local semiflow $\mathcal{F}_k = \{\Sigma_k, \mathcal{G}_k\}$, $k = 1, \ldots, K-1$, $\mathbf{x}^{(k)}(t) = \mathbf{\Phi}^{(k)}_{t,t_{k-1}}(\mathbf{x}^{(k)}(t_{k-1}))$ is uniformly continuous with respect to the initial state $\mathbf{x}^{(k)}(t_{k-1})$ on $\bar{I}_k$. Moreover, $\|\mathbf{x}^{(k)}(t)\|^{(k)}$ is uniformly bounded on $\bar{I}_k$. Thus, the positive semi-orbits of $\mathcal{F}$ do not escape to infinity in finite time. This leads to the following result:

Theorem 3.1: A necessary and sufficient condition for a positively invariant set Γ_1 of a switched semiflow $\mathcal{F} = \{\Sigma_1, \mathcal{G}\}$ to be stable (respectively, asymptotically stable) in the sense of Lyapunov is that the positively invariant set Γ_K of the last local semiflow $\mathcal{F}_K = \{\Sigma_K, \mathcal{G}_K\}$ is stable (respectively, asymptotically stable) with respect to the norm of Σ_K.

Proof: Sufficiency: Assume that the positively invariant set Γ_K of $\mathcal{F}_K$ is stable. Then, for any $\epsilon > 0$, there exists a $\delta_K > 0$ such that

$$\mathbf{x}^{(K)}(t_{K-1}) \in \eta_{\delta_K}(\Gamma_K) \Rightarrow \rho_K(\mathbf{\Phi}^{(K)}_{t,t_{K-1}}(\mathbf{x}^{(K)}(t_{K-1})), \Gamma_K) < \epsilon \quad \text{for all } t \geq t_{K-1} \tag{6}$$

where $\eta_{\delta_k}(\Gamma_k)$ denotes the δ_k neighborhood of the positively invariant set Γ_k of $\mathcal{F}_k$ defined by

$$\eta_{\delta_k}(\Gamma_k) = \{\mathbf{x}^{(k)} \in \Sigma_k\colon \rho_k(\mathbf{x}^{(k)}, \Gamma_k) < \delta_k\} \tag{7}$$

Consider the family of evolution operators $\mathbf{\Phi}_{t,t_0}$ defined by

$$\mathbf{\Phi}_{t,t_0} = \mathbf{\Phi}^{(k)}_{t,t_{k-1}} \circ \left(\mathbf{S}^{(k,k-1)} \circ \mathbf{\Phi}^{(k-1)}_{t_{k-1},t_{k-2}}\right) \circ \cdots \circ \left(\mathbf{S}^{(2,1)} \circ \mathbf{\Phi}^{(1)}_{t_1,t_0}\right), \qquad k = 1, \ldots, K-1 \tag{8}$$

From the assumption that $\mathbf{\Phi}^{(k)}_{t,t_{k-1}}$ is uniformly continuous on $\bar{I}_k$, and $\mathbf{S}^{(k,k-1)}$ is a continuous mapping on $\Sigma_{k-1} \to \Sigma_k$, $\mathbf{\Phi}_{t,t_0}$ is uniformly continuous on $[t_0, t] \subseteq \bigcup_{i=1}^{K-1} \bar{I}_i$, that is, for any $\tilde{\epsilon} > 0$, there exists a $\tilde{\delta} > 0$ such that

$$\mathbf{x}^{(1)} \in \eta_{\tilde{\delta}}(\Gamma_1) \Rightarrow \rho_k(\mathbf{\Phi}_{t,t_0}(\mathbf{x}^{(1)}), \Gamma_1) < \tilde{\epsilon} \quad \text{for all } t \in \bar{I}_k, \qquad k = 1, \ldots, K-1 \tag{9}$$

Because $\Gamma_K = \mathbf{S}^{(K,K-1)}(\Gamma_{K-1})$, then $\mathbf{S}^{(K,K-1)}(\eta_{\tilde{\epsilon}}(\Gamma_{K-1})) \subseteq \eta_{\delta_K}(\Gamma_K)$ for some $\tilde{\epsilon} > 0$. If $\tilde{\epsilon} > \epsilon$, then it is always possible to choose a $\tilde{\delta}_K < \delta_K$ such that $\mathbf{S}^{(K,K-1)}(\eta_\epsilon(\Gamma_{K-1})) \subseteq \eta_{\tilde{\delta}_K}(\Gamma_K)$. Thus, for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\mathbf{x}^{(1)} \in \eta_\delta(\Gamma_1) \Rightarrow \rho_k\Big(\Big(\mathbf{\Phi}^{(K)}_{t,t_{K-1}} \circ (\mathbf{S}^{(K,K-1)} \circ \mathbf{\Phi}_{t_{K-1},t_0})\Big)(\mathbf{x}^{(1)}), \Gamma_K\Big) < \epsilon \quad \text{for all } t \geq t_{K-1} \tag{10}$$

implying that Γ_1 is stable. Evidently, if Γ_K is asymptotically stable, so is Γ_1.

Necessity: Obvious. □

B. Finite-Time Stability with Specified Bounds

In aerospace applications in which the control actions take place over finite-time durations, it is useful to define a special form of finite-time stability. For example, in space missions involving multiple spacecraft with docking and undocking capabilities flying in formation over a given finite control time duration, docking and/or undocking occurs at specified time instants. It is important that the deviations of the geometric formation pattern of the spacecraft from the desired one in terms of some metrics during and at the end of the control time duration are within specified bounds. The latter bound is generally more stringent than that for the entire control time duration. Thus, it is useful to introduce the notion of finite-time stability with specified bounds for switched dynamical systems with state-space dilation and contraction. This notion differs from those introduced earlier by Chatterjee and Liberzon [12,13].

Consider a system in the form of Eq. (1) with state space Σ_k defined on the specified time interval $I_k = [t_{k-1}, t_k]$.

Definition 3.5: A positively invariant set Γ_k of the local semiflow $\mathcal{F}_k$ generated by Eq. (1) is said to be finite-time stable with specified bounds ϵ_k and μ_k with $\epsilon_k > \mu_k > 0$ if there exists a δ neighborhood $\eta_\delta(\Gamma_k) \subset \Sigma_k$ such that for every $\mathbf{x}^{(k)}(t_{k-1}) \in \eta_\delta(\Gamma_k)$, its corresponding local positive semi-orbit satisfies

$$\sup_{t \in I_k} \rho_k(\mathbf{\Phi}_t(\mathbf{x}^{(k)}(t_{k-1})), \Gamma_k) < \epsilon_k \tag{11}$$

and

$$\rho_k(\mathbf{\Phi}_{t_k}(\mathbf{x}^{(k)}(t_{k-1})), \Gamma_k) \leq \mu_k \tag{12}$$

Definition 3.6: A positively invariant set $\Gamma_1 \subset \Sigma_1$ of the switched semiflow $\mathcal{F}$ defined on $T_K = \bigcup_{k=1}^K I_k$ with $t_K < \infty$ is said to be finite-time stable with specified bounds ϵ and μ with $\epsilon > \mu$ if there exists a δ neighborhood $\eta_\delta(\Gamma_1) \subset \Sigma_1$ such that for every $\mathbf{x}^{(1)}(t_0) \in \eta_\delta(\Gamma_1)$, its corresponding positive semi-orbit satisfies

$$\max\{\sup_{t \in I_k} \rho_k(\mathbf{\Phi}_{t,t_{k-1}}(\mathbf{x}^{(k)}(t_{k-1})), \Gamma_k), k = 1, \ldots, K\} < \epsilon \tag{13}$$

and

$$\rho_K(\mathbf{\Phi}_{t_K,t_{K-1}}(\mathbf{x}^{(K)}(t_{K-1})), \Gamma_K) \leq \mu \tag{14}$$

Inequality (13) corresponds to a specified uniform bound of the positive semi-orbit over T_K, whereas inequality (14) represents a specified bound on the terminal state at time t_K. In some applications, it is useful to specify a bound ϵ_k for each subinterval I_k, and replace (13) by $\sup_{t \in I_k} \rho_k(\mathbf{\Phi}_{t,t_{k-1}}(\mathbf{x}^{(k)}(t_{k-1})), \Gamma_k) < \epsilon_k$, $k = 1, \ldots, K$.

IV. Stability Conditions

In what follows, we shall derive stability conditions for an important special class of switched semidynamical systems or semiflows formed by concatenating a set of local semiflows described by

$$\mathrm{d}\,\mathbf{x}^{(k)}/\mathrm{d}t = \mathbf{A}_k \mathbf{x}^{(k)} + \mathbf{g}^{(k)}(t, \mathbf{x}^{(k)}), \qquad k = 1, \ldots, K \tag{15}$$

defined on the state space $\Sigma_k = \mathbb{R}^{n_k}$ (embedded in the complex coordinate space $\mathbb{C}^{n_k}$ if necessary) with state-concatenation condition

$$\mathbf{x}^{(k)}(t_{k-1}) = \mathbf{S}^{(k,k-1)}(\mathbf{x}^{(k-1)}(t_{k-1})), \qquad k = 2, \ldots, K \tag{16}$$

where $\mathbf{A}_k$ is a specified $n_k \times n_k$ nonsingular constant real matrix, and $\mathbf{g}^{(k)} \to \mathbf{g}^{(k)}(t, \mathbf{x}^{(k)})$ from $\bar{I}_k \times \mathbb{R}^{n_k} \to \mathbb{R}^{n_k}$ is a given function satisfying the following global Lipschitz condition about the origin of $\mathbb{R}^{n_k}$:

$$\|\mathbf{g}^{(k)}(t, \mathbf{x}^{(k)})\|^{(k)} \leq \gamma_k(t)\|\mathbf{x}^{(k)}\|^{(k)} \quad \text{for all } (t, \mathbf{x}^{(k)}) \in \bar{I}_k \times \mathbb{R}^{n_k} \tag{17}$$

where $\|\cdot\|^{(k)}$ is a suitable norm on Σ_k, and $\gamma_k = \gamma_k(t)$ is a positive real-valued locally integrable function defined on $\bar{I}_k$. For simplicity, we assume that the concatenation operator $\mathbf{S}^{(k,k-1)}$ is a linear transformation on $\mathbb{R}^{n_{k-1}} \to \mathbb{R}^{n_k}$.

Remark 4.1: Under the assumption that $\mathbf{A}_k$, $k = 1, \ldots, K$ are nonsingular, we have

$$\|\mathbf{x}^{(k)}\|^{(k)} \leq \|\mathbf{A}_k^{-1}\|^{(k)} \|\mathbf{g}^{(k)}(t, \mathbf{x}^{(k)})\|^{(k)} \tag{18}$$

Evidently, the origin of $\mathbb{R}^{n_k}$ is the unique equilibrium state for the kth elemental system, because if there exists a $\mathbf{x}^{(k)} \neq \mathbf{0}$ such that $\mathbf{g}^{(k)}(t, \mathbf{x}^{(k)}) = \mathbf{0}$ for all $t \in \bar{I}_k$, estimate (18) leads to a contradiction.

To derive stability conditions, we need the following estimate for the solutions of Eq. (15), which can be readily deduced from Eq. (17) and the Gronwall–Bellman inequality.

Lemma 4.1: Assume that $\mathbf{g}^{(k)}$ satisfies Eq. (17). Then any solution $\mathbf{x}^{(k)}(t)$ of Eq. (15) with the initial state $\mathbf{x}^{(k)}(t_{k-1})$ satisfies the estimate

$$\|\mathbf{x}^{(k)}(t)\|^{(k)} \leq \|\mathbf{x}^{(k)}(t_{k-1})\|^{(k)} \exp\bigg(\beta_k(t - t_{k-1}) + \int_{t_{k-1}}^t \alpha_k \gamma_k(\tau)\,\mathrm{d}\tau\bigg) \quad \text{for all } t \in \bar{I}_k \tag{19}$$

where α_k and β_k are the parameters in the estimate

$$\|\exp[\mathbf{A}_k(t - t_{k-1})]\|^{(k)} \leq \alpha_k \exp[\beta_k(t - t_{k-1})] \quad \text{for } t \in I_k \tag{20}$$

Remark 4.2: The assumption that the state concatenation operator $\mathbf{S}^{(k,k-1)}$ is a linear transformation on $\mathbb{R}^{n_{k-1}} \to \mathbb{R}^{n_k}$ may be replaced by the weaker condition that $\mathbf{S}^{(k,k-1)}$ is a continuous mapping on $\mathbb{R}^{n_{k-1}} \to \mathbb{R}^{n_k}$, such that $\mathbf{S}^{(k,k-1)}$ maps the origin of $\mathbb{R}^{n(k-1)}$ into the origin of $\mathbb{R}^{n_k}$.

Remark 4.3: The choice of the norm $\|\cdot\|^{(k)}$ for Σ_k depends on the specific system under consideration. In many aerospace applications, the generalized Euclidean norm $\|\mathbf{x}^{(k)}\|^{(k)} = \{(\mathbf{x}^{(k)})^T \mathbf{P}_k \mathbf{x}^{(k)}\}^{1/2}$ is a suitable choice, where $\mathbf{P}_k$ is a positive-definite real symmetric matrix.

By applying Lemma 4.1 recursively, we obtain the following estimate for any positive semi-orbit of the switched semiflow generated by Eqs. (15) and (16):

Lemma 4.2: Assume that $\mathbf{g}^{(k)}$, $k = 1, \ldots, K$ satisfy Eq. (17). Then any positive semi-orbit of the switched semiflow generated by Eqs. (15) and (16) satisfies the estimate

$$\|\mathbf{x}^{(k)}(t)\|^{(k)} \leq \bigg(\prod_{i=1}^k \alpha_i\bigg)\bigg(\prod_{j=2}^k \|\mathbf{S}^{(j,j-1)}\|^{(j,j-1)}\bigg) \exp\bigg[\bigg(\beta_k + \frac{\alpha_k}{(t - t_{k-1})}\int_{t_{k-1}}^t \gamma_k(\tau)\,\mathrm{d}\tau\bigg)(t - t_{k-1}) + \sum_{i=1}^{k-1}(\beta_i + \alpha_i\bar{\gamma}_i)(t_i - t_{i-1})\bigg]\|\mathbf{x}^{(1)}(t_0)\|^{(1)} \quad \text{for all } t \in I_k \tag{21}$$

where

$$\bar{\gamma}_i = \frac{1}{(t_i - t_{i-1})}\int_{t_{i-1}}^{t_i} \gamma_i(\tau)\,\mathrm{d}\tau \tag{22}$$

and

$$\|\mathbf{S}^{(i,i-1)}\|^{(i,i-1)} \overset{\mathrm{def}}{=} \sup\{\|\mathbf{S}^{(i,i-1)}(\mathbf{x}^{(i-1)})\|^{(i)}\colon \|\mathbf{x}^{(i-1)}\|^{(i-1)} = 1\} \tag{23}$$

In what follows, we shall give sufficient conditions for Lyapunov stability and finite-time stability with specified bounds.

Theorem 4.1: Assume $K \geq 2$. The zero state of the switched semiflow generated by Eqs. (15) and (16) is asymptotically stable in the sense of Lyapunov if

$$-\alpha_K \bar{\gamma}_K < \beta_K < 0 \tag{24}$$

where

$$\bar{\gamma}_K \stackrel{\text{def}}{=} \lim_{t \to t_K} \frac{1}{(t - t_{K-1})} \int_{t_{K-1}}^{t} \gamma_K(\tau)\,\mathrm{d}\tau \tag{25}$$

Proof: Because the concatenation operators $\mathbf{S}^{(k,k-1)}$, $k = 1, \ldots, K$ are linear transformations, the zero equilibrium state of the first elemental system is mapped to the zero equilibrium state of the last or Kth elemental system. From Theorem 3.1, the asymptotic stability of the zero equilibrium state of the Kth elemental system implies the asymptotic stability of the zero equilibrium state of the switched semidynamical system. From Lemma 3.1, it is evident that the asymptotic stability of the zero equilibrium state of the Kth elemental system is ensured under condition (24). □

Remark 4.4: The parameter $\bar{\gamma}_K$ corresponds to the average value of $\gamma_K(\cdot)$ over the time interval I_K, where γ_K may be a time-varying parameter associated with the disturbance bound (17).

Theorem 4.2: Assume $K \geq 2$ and $t_K < \infty$. Given positive real numbers μ and ϵ with $\mu < \epsilon$, if

$$\|\mathbf{x}^{(1)}(t_0)\|^{(1)} \leq \min\left\{\frac{\epsilon}{\sigma}, \frac{\mu}{\kappa}\right\} \tag{26}$$

where

$$\begin{aligned}\sigma \stackrel{\text{def}}{=} \max\Bigg\{ \sup_{t \in I_k} \Bigg\{ \left(\prod_{i=1}^{k} \alpha_i\right) \left(\prod_{j=2}^{k} \|\mathbf{S}^{(j,j-1)}\|^{(j,j-1)}\right) \exp\Bigg[\Bigg(\beta_k \\ + \frac{\alpha_k}{(t - t_{k-1})} \int_{t_{k-1}}^{t} \gamma_k(\tau)\,\mathrm{d}\tau\Bigg)(t - t_{k-1}) \\ + \sum_{i=1}^{k-1} (\beta_i + \alpha_i \bar{\gamma}_i)(t_i - t_{i-1})\Bigg]\Bigg\}, k = 1, \ldots, k \Bigg\}\end{aligned} \tag{27}$$

and

$$\kappa \stackrel{\text{def}}{=} \left(\prod_{i=1}^{K} \alpha_i\right) \left(\prod_{j=2}^{K} \|\mathbf{S}^{(j,j-1)}\|^{(j,j-1)}\right) \exp\left[\sum_{k=1}^{K} (\beta_k + \alpha_k)(t_k - t_{k-1})\right] \tag{28}$$

then the zero state of the switched semidynamical system is finite-time stable with specified bounds μ and ϵ.

Proof: From Lemma 4.2, we have

$$\begin{aligned}\|\mathbf{x}^{(k)}(t)\|^{(k)} \leq \left(\prod_{i=1}^{k} \alpha_i\right) \left(\prod_{j=2}^{k} \|\mathbf{S}^{(j,j-1)}\|^{(j,j-1)}\right) \exp\Bigg[\Bigg(\beta_k \\ + \frac{\alpha_k}{(t - t_{k-1})} \int_{t_{k-1}}^{t} \gamma_k(\tau)\,\mathrm{d}\tau\Bigg)(t - t_{k-1}) \\ + \sum_{i=1}^{k-1} (\beta_i + \alpha_i \bar{\gamma}_i)(t_i - t_{i-1})\Bigg] \|\mathbf{x}^{(1)}(t_0)\|^{(1)} \leq \sigma \|\mathbf{x}^{(1)}(t_0)\|^{(1)} \\ \text{for all } t \in I_k, \qquad k = 1, \ldots, K\end{aligned} \tag{29}$$

Let $\|\mathbf{x}^{(1)}(t_0)\|^{(1)} \leq \delta = \min\{\epsilon/\sigma, \mu/\kappa\}$. It follows that

$$\|\mathbf{x}^{(k)}(t)\|^{(k)} \leq \sigma\delta \leq \epsilon \quad \text{for all } t \in I_k, \qquad k = 1, \ldots, K \tag{30}$$

and

$$\|\mathbf{x}^{(K)}(t_K)\|^{(K)} \leq \kappa\delta \leq \mu \tag{31}$$

Thus, conditions (13) and (14) with $\Gamma_k = \{\mathbf{0}\}$, $k = 1, \ldots, K$ are both satisfied. By definition, we have finite-time stability with specified bounds ϵ and μ. □

Remark 4.5: The foregoing results can be extended to the case in which condition (17) is replaced by

$$\begin{aligned}\|\mathbf{g}^{(k)}(t, \mathbf{x}^{(k)})\|^{(k)} \leq \gamma_k^o(t) + \gamma_k(t)\|\mathbf{x}^{(k)}\|^{(k)} \\ \text{for all } (t, \mathbf{x}^{(k)}) \in \bar{I}_k \times \mathbb{R}^{n_k}\end{aligned} \tag{32}$$

where γ_k^o and γ_k are specified real-valued nonnegative locally integrable functions of t defined on $\bar{I}_k$.

Remark 4.6: In the case in which system equation (15) is replaced by

$$\mathrm{d}\,\mathbf{x}^{(k)}/\mathrm{d}t = \tilde{\mathbf{A}}_k \mathbf{x}^{(k)} + \mathbf{B}_k \mathbf{u}^{(k)} + \mathbf{g}^{(k)}(t, \mathbf{x}^{(k)}), \qquad k = 1, \ldots, K \tag{33}$$

where $\mathbf{u}^{(k)}$, $k = 1, \ldots, K$ correspond to the controls; if $(\tilde{\mathbf{A}}_k, \mathbf{B}_k)$, $k = 1, \ldots, K$ are controllable, then one can always find suitable linear state-feedback controls of the form $\mathbf{u}^{(k)} = \mathbf{K}_k \mathbf{x}^{(k)}$ such that $\mathbf{A}_k \stackrel{\text{def}}{=} (\tilde{\mathbf{A}}_k + \mathbf{B}_k \mathbf{K}_k)$ is stable and $\|\exp[\mathbf{A}_k(t - t_{k-1})]\|^{(k)}$ has an exponential bound of the form equation (20) with sufficiently negative β_k to satisfy condition (24) or (26).

Remark 4.7: The stability definitions 3.4 and 3.5 and the stability conditions given in Proposition 3.1 and Theorems 3.1 and 3.2 remain valid for the special case in which there is no state-space dilation or contraction. In this case, although $\dim(\Sigma_{k-1}) = \dim(\Sigma_k)$ and the systems defined on the time intervals I_{k-1} and I_k are different, the norms for Σ_{k-1} and Σ_k may not be the same. Moreover, the concatenation operator $\mathbf{S}^{(k,k-1)}$ on $\Sigma_{k-1} \to \Sigma_k$ may not be invertible.

V. Application to Spacecraft Formation Stability Analysis

Consider a formation of four spacecraft (labeled 1–4) consisting of a reference spacecraft (no. 1), two identical basic spacecraft (nos. 2 and 3) with a single docking port, and a spacecraft (no. 4) with two docking ports as shown in Fig. 1. Initially, at time $t_0 = 0$, the mass centers of spacecraft 1, 2, and 3 are at the vertices of an equilateral triangle, and the mass center of spacecraft 4 is located at the midpoint of the line connecting the mass centers of spacecraft 2 and 3. At a specified time $t_1 > 0$, spacecraft 2 and 4 dock with spacecraft 3 simultaneously to form an aggregated spacecraft 2′ that tries to maintain a fixed distance from the reference spacecraft 1.

Let $\mathcal{F}_o$ denote the inertial frame with origin O in the three-dimensional real Euclidean space $\mathbb{E}^3$ with orthonormal basis $\mathcal{B}_o = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then any position vector $\mathbf{r} \in \mathbb{E}^3$ can be represented as

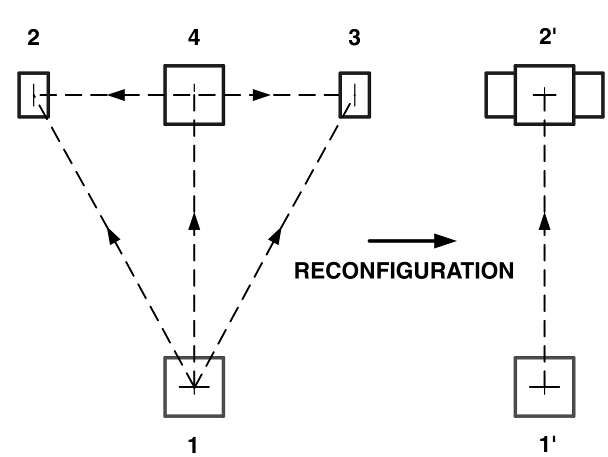

Fig. 1 Reconfiguration of a four-spacecraft formation involving docking. Arrows specify the directions of signal flow between the spacecraft.

$$\mathbf{r} = \sum_{i=1}^{3} r_i \mathbf{e}_i$$

where $[\mathbf{r}]_o = [r_1, r_2, r_3]^T$ is the representation of $\mathbf{r}$ with respect to $\mathcal{B}_o$. We assume that each spacecraft can be represented by a rigid body with mass M_i, whose mass center at time t is denoted by $\mathbf{r}_i(t)$. Let $\mathbf{r}_i(t)$ and $\mathbf{r}_i^d(t)$ represent, respectively, the actual and desired positions of the mass center at time t relative to $\mathcal{F}_o$ corresponding to the ith spacecraft. We define the actual formation pattern at time $t < t_1$ as a point set $\mathcal{P}(t) = \{\mathbf{r}_1(t), \mathbf{r}_2(t), \mathbf{r}_3(t), \mathbf{r}_4(t)\}$ and the desired formation pattern at time t as $\mathcal{P}_d(t) = \{\mathbf{r}_1^d(t), \mathbf{r}_2^d(t), \mathbf{r}_3^d(t), \mathbf{r}_4^d(t)\}$. We assume that the desired formation patterns are shape invariant, that is, the Euclidean distance between any pair of distinct points in the desired formation pattern set is constant. Before docking takes place, the translational motion of the spacecraft with respect to $\mathcal{F}_o$ can be described by

$$M_i \ddot{\mathbf{r}}_i(t) = \mathbf{u}_i(t) + \boldsymbol{\eta}_i, \qquad i = 1, \ldots, 4 \tag{34}$$

where $\mathbf{u}_i$ denotes the control force, and $\boldsymbol{\eta}_i$ represents an external disturbance force. We assume that simultaneous docking of spacecraft 2 and 3 with spacecraft 4 at $t = t_1$ can be accomplished. Once simultaneous docking of these spacecraft is achieved, the motion of the aggregated spacecraft $2'$ can be described by

$$M_{2'} \ddot{\mathbf{r}}_{2'}(t) = \mathbf{u}_{2'}(t) + \boldsymbol{\eta}_{2'}, \qquad M_{2'} = M_2 + M_3 + M_4 \tag{35}$$

where $\mathbf{r}_{2'}$ represents the mass center of the aggregated spacecraft, and $\mathbf{u}_{2'}$ and $\boldsymbol{\eta}_{2'}$ denote the control force and disturbance, respectively. Assuming that the linear momentum is conserved (no transfer of linear momentum into angular momentum) before and after docking, we have

$$M_{2'} \dot{\mathbf{r}}_{2'}\left(t_1^+\right) = M_2 \dot{\mathbf{r}}_2\left(t_1^-\right) + M_3 \dot{\mathbf{r}}_3\left(t_1^-\right) + M_4 \dot{\mathbf{r}}_4\left(t_1^-\right) \tag{36}$$

where t_1^- and t_1^+ denote the time instants right before and after the simultaneous docking, respectively. Also, from conservation of the center of mass of the spacecraft before and after docking, we have

$$M_{2'} \mathbf{r}_{2'}\left(t_1^+\right) = M_2 \mathbf{r}_2\left(t_1^-\right) + M_3 \mathbf{r}_3\left(t_1^-\right) + M_4 \mathbf{r}_4\left(t_1^-\right) \tag{37}$$

Similar conditions can be established for the positions and velocities of the spacecraft when an aggregated spacecraft breaks up into basic spacecraft through undocking.

For simplicity, we assume that proportional-plus derivative feedback controls are used for both automatic docking and formation alignment. During the docking maneuver period $I_1 = [0, t_1]$, the control laws for all spacecraft have the form

$$\mathbf{u}_i = \mathbf{K}^{(P,i)} \mathbf{E}_i + \mathbf{K}^{(R,i)} \dot{\mathbf{E}}_i, \qquad i = 1, 2, 3, 4 \tag{38}$$

where $\mathbf{K}^{(P,i)}$ and $\mathbf{K}^{(R,i)}$ are diagonal feedback gain matrices with positive diagonal elements $K_{jj}^{(P,i)}$ and $K_{jj}^{(R,i)}$, $j = 1, 2, 3$, respectively; and $\mathbf{E}_i$ and $\dot{\mathbf{E}}_i$ are error and error rates defined by

$$\mathbf{E}_1(t) = \mathbf{r}_1^d(t) - \mathbf{r}_1(t), \qquad \dot{\mathbf{E}}_1(t) = \dot{\mathbf{r}}_1^d(t) - \dot{\mathbf{r}}_1(t) \tag{39}$$

$$\begin{aligned} \mathbf{E}_i(t) &= \mathbf{r}_4(t) + \boldsymbol{\Delta}_{i,4} - \mathbf{r}_i(t) \\ \dot{\mathbf{E}}_i(t) &= \dot{\mathbf{r}}_4(t) - \dot{\mathbf{r}}_i(t), \qquad i = 2, 3 \end{aligned} \tag{40}$$

$$\mathbf{E}_4(t) = \mathbf{r}_1(t) + \boldsymbol{\Delta}_{4,1} - \mathbf{r}_4(t), \qquad \dot{\mathbf{E}}_4(t) = \dot{\mathbf{r}}_1(t) - \dot{\mathbf{r}}_4(t) \tag{41}$$

where $\boldsymbol{\Delta}_{i,j}$ is a nonzero constant vector in $\mathbb{E}^3$ specifying the desired separation between the mass centers of the ith and jth spacecraft after docking has occurred, and $\mathbf{r}_1^d = \mathbf{r}_1^d(t)$ is a twice continuously differentiable function of t specifying the desired position for spacecraft 1 with respect to the inertial frame. In what follows, we shall drop the bracket notation $[\cdot]_o$ for brevity when ambiguity does not arise (e.g., $[\mathbf{r}]_o$ will be denoted simply by $\mathbf{r}$).

Let the error state of the formation at any time $t \in I_1$ be defined by $\mathbf{x}^{(1)}(t) \stackrel{\text{def}}{=} [\mathbf{E}_1^T, \dot{\mathbf{E}}_1^T, \mathbf{E}_2^T, \dot{\mathbf{E}}_2^T, \mathbf{E}_3^T, \dot{\mathbf{E}}_3^T, \mathbf{E}_4^T, \dot{\mathbf{E}}_4^T]^T(t)$, where $\mathbf{E}_i(t) = [E_{i1}, E_{i2}, E_{i3}]^T(t)$. We assume that the disturbances $\boldsymbol{\eta}_i$ depend on certain combinations of $\mathbf{E}_j$, $\dot{\mathbf{E}}_j$, $j = 1, \ldots, 4$, and satisfy the following bounds:

$$\|\boldsymbol{\eta}_1(t, \mathbf{E}_1, \dot{\mathbf{E}}_1)\|_{\mathbb{R}^3} \leq \kappa_1(t) \|(\mathbf{E}_1, \dot{\mathbf{E}}_1)\|_{\mathbb{R}^6} \tag{42}$$

$$\begin{aligned} &\|\boldsymbol{\eta}_i(t, \mathbf{E}_1, \mathbf{E}_i, \mathbf{E}_4, \dot{\mathbf{E}}_1, \dot{\mathbf{E}}_i, \dot{\mathbf{E}}_4)\|_{\mathbb{R}^3} \\ &\quad \leq \kappa_i(t) \|(\mathbf{E}_1, \mathbf{E}_i, \mathbf{E}_4, \dot{\mathbf{E}}_1, \dot{\mathbf{E}}_i, \dot{\mathbf{E}}_4)\|_{\mathbb{R}^{18}} \qquad i = 2, 3 \end{aligned} \tag{43}$$

$$\|\boldsymbol{\eta}_4(t, \mathbf{E}_1, \mathbf{E}_4, \dot{\mathbf{E}}_1, \dot{\mathbf{E}}_4)\|_{\mathbb{R}^3} \leq \kappa_4(t) \|(\mathbf{E}_1, \mathbf{E}_4, \dot{\mathbf{E}}_1, \dot{\mathbf{E}}_4)\|_{\mathbb{R}^{12}} \tag{44}$$

for all $t \in \bar{I}_1$, where $\kappa_i = \kappa_i(t)$, $i = 1, \ldots, 4$ are specified real-valued nonnegative locally integrable functions of t defined on $\bar{I}_1$, and $\|\cdot\|_{\mathbb{R}^n}$ denotes a suitable norm for $\mathbb{R}^n$. For spacecraft formation stability analysis, it is useful to choose the ℓ_∞-norm (i.e., $\|\mathbf{x}\|_{\mathbb{R}^n} = \max\{|x_1|, \ldots, |x_n|\}$, where $\mathbf{x} = [x_1, \ldots, x_n]^T$). It can be deduced from Eqs. (34) and (39–41) that the differential equation for $\mathbf{x}^{(1)}$ with feedback controls (38) can be written as

$$\frac{\mathrm{d}\mathbf{x}^{(1)}}{\mathrm{d}t} = \mathbf{A}_1 \mathbf{x}^{(1)} + \mathbf{g}^{(1)}(t, \mathbf{x}^{(1)}) \tag{45}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{O}_{6,6} & \mathbf{O}_{6,6} & \mathbf{O}_{6,6} \\ \mathbf{O}_{6,6} & \mathbf{Q}_2 & \mathbf{O}_{6,6} & \mathbf{W}_4 \\ \mathbf{O}_{6,6} & \mathbf{O}_{6,6} & \mathbf{Q}_3 & \mathbf{W}_4 \\ \mathbf{W}_1 & \mathbf{O}_{6,6} & \mathbf{O}_{6,6} & \mathbf{Q}_4 \end{bmatrix}$$

$$\mathbf{g}^{(1)} = \begin{bmatrix} \mathbf{0}_3 \\ \ddot{\mathbf{r}}_1^d(t) - \hat{\boldsymbol{\eta}}_1(t, \mathbf{E}_1, \dot{\mathbf{E}}_1) \\ \mathbf{0}_3 \\ \hat{\boldsymbol{\eta}}_4(t, \mathbf{E}_1, \mathbf{E}_4, \dot{\mathbf{E}}_1, \dot{\mathbf{E}}_4) - \hat{\boldsymbol{\eta}}_2(t, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_4, \dot{\mathbf{E}}_1, \dot{\mathbf{E}}_2, \dot{\mathbf{E}}_4) \\ \mathbf{0}_3 \\ \hat{\boldsymbol{\eta}}_4(t, \mathbf{E}_1, \mathbf{E}_4, \dot{\mathbf{E}}_1, \dot{\mathbf{E}}_4) - \hat{\boldsymbol{\eta}}_3(t, \mathbf{E}_1, \mathbf{E}_3, \mathbf{E}_4, \dot{\mathbf{E}}_1, \dot{\mathbf{E}}_3, \dot{\mathbf{E}}_4) \\ \mathbf{0}_3 \\ \hat{\boldsymbol{\eta}}_1(t, \mathbf{E}_1, \dot{\mathbf{E}}_1) - \hat{\boldsymbol{\eta}}_4(t, \mathbf{E}_1, \mathbf{E}_4, \dot{\mathbf{E}}_1, \dot{\mathbf{E}}_4) \end{bmatrix} \tag{46}$$

with $\hat{\boldsymbol{\eta}}_i \stackrel{\text{def}}{=} \boldsymbol{\eta}_i / M_i$, $i = 1, \ldots, 4$, and

$$\begin{aligned} \mathbf{Q}_i &= \begin{bmatrix} \mathbf{O}_{3,3} & \mathbf{I}_3 \\ -\mathbf{K}^{(P,i)}/M_i & -\mathbf{K}^{(R,i)}/M_i \end{bmatrix}, \qquad i = 1, \ldots, 4 \\ \mathbf{W}_j &= \begin{bmatrix} \mathbf{O}_{3,3} & \mathbf{O}_{3,3} \\ \mathbf{K}^{(P,j)}/M_j & \mathbf{K}^{(R,j)}/M_j \end{bmatrix}, \qquad j = 1, 4 \end{aligned} \tag{47}$$

where $\mathbf{O}_{i,j}$ and $\mathbf{I}_i$ are $i \times j$ zero and $i \times i$ identity matrices, respectively, and $\mathbf{0}_3 = [0, 0, 0]^T$.

After simultaneous docking of spacecraft 2 and 3 with spacecraft 4 to form the aggregated spacecraft $2'$ at $t = t_1$, the two-spacecraft formation error state at any time $t \in \bar{I}_2 = [t_1, t_2]$ can be described by $\mathbf{x}^{(2)}(t) \stackrel{\text{def}}{=} [\mathbf{E}_{1'}(t)^T, \dot{\mathbf{E}}_{1'}(t)^T, \mathbf{E}_{2'}(t)^T, \dot{\mathbf{E}}_{2'}(t)^T]^T$, where $\mathbf{E}_{1'} = \mathbf{E}_1$, $\dot{\mathbf{E}}_{1'} = \dot{\mathbf{E}}_1$ as defined in Eq. (39), and

$$\mathbf{E}_{2'}(t) = \mathbf{r}_1(t) + \boldsymbol{\Delta}_{2',1} - \mathbf{r}_{2'}(t), \qquad \dot{\mathbf{E}}_{2'}(t) = \dot{\mathbf{r}}_1(t) - \dot{\mathbf{r}}_{2'}(t) \tag{48}$$

Let the feedback control for reference spacecraft $1'$ be given as before by Eq. (38), and the feedback control for the aggregated Spacecraft $2'$ be given by

$$\mathbf{u}_{2'} = \mathbf{K}^{(P,2')} \mathbf{E}_{2'} + \mathbf{K}^{(R,2')} \dot{\mathbf{E}}_{2'} \tag{49}$$

Then the differential equation for $\mathbf{x}^{(2)}(t)$ with feedback controls (38) and (49) is given by

$$\frac{\mathrm{d}\mathbf{x}^{(2)}}{\mathrm{d}t} = \mathbf{A}_2\mathbf{x}^{(2)} + \mathbf{g}^{(2)}(t, \mathbf{x}^{(2)}) \tag{50}$$

where

$$\mathbf{A}_2 = \begin{bmatrix} \mathbf{Q}_{1'} & \mathbf{O}_{6,6} \\ \mathbf{W}_1 & \mathbf{Q}_{2'} \end{bmatrix}$$
$$\mathbf{g}^{(2)} = \begin{bmatrix} \mathbf{0}_3 \\ \ddot{\mathbf{r}}_1^d(t) - \hat{\boldsymbol{\eta}}_{1'}(t, \mathbf{E}_{1'}, \dot{\mathbf{E}}_{1'}) \\ \mathbf{0}_3 \\ \hat{\boldsymbol{\eta}}_{1'}(t, \mathbf{E}_{1'}, \dot{\mathbf{E}}_{1'}) - \hat{\boldsymbol{\eta}}_{2'}(t, \mathbf{E}_{1'}, \mathbf{E}_{2'}, \dot{\mathbf{E}}_{1'}, \dot{\mathbf{E}}_{2'}) \end{bmatrix} \tag{51}$$

with $\hat{\boldsymbol{\eta}}_{i'} \overset{\text{def}}{=} \boldsymbol{\eta}_{i'}/M_{i'}$, $i' = 1', 2'$. Note that because the spacecraft before and after docking have different geometric shapes, $\boldsymbol{\eta}_{2'} \neq \boldsymbol{\eta}_2$. We assume that $\boldsymbol{\eta}_{1'}$ and $\boldsymbol{\eta}_{2'}$ satisfy the following bounds:

$$\|\boldsymbol{\eta}_{1'}(t, \mathbf{E}_{1'}, \dot{\mathbf{E}}_{1'})\|_{\mathbb{R}^3} \leq \kappa_{1'}(t)\|(\mathbf{E}_{1'}, \dot{\mathbf{E}}_{1'})\|_{\mathbb{R}^6} \tag{52}$$

$$\|\boldsymbol{\eta}_{2'}(t, \mathbf{E}_{1'}, \mathbf{E}_{2'}, \dot{\mathbf{E}}_{1'}, \dot{\mathbf{E}}_{2'})\|_{\mathbb{R}^3} \leq \kappa_{2'}(t)\|(\mathbf{E}_{1'}, \mathbf{E}_{2'}, \dot{\mathbf{E}}_{1'}, \dot{\mathbf{E}}_{2'})\|_{\mathbb{R}^{12}} \tag{53}$$

where $\kappa_i = \kappa_i(t)$, $i = 1', 2'$ are specified real-valued nonnegative locally integrable functions of t defined on $\bar{I}_2$. The operator concatenating the states $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ at time t_1, in view of Eqs. (36) and (37), is given by

$$\mathbf{S}^{(2,1)} = \begin{bmatrix} \mathbf{I}_6 & & & \mathbf{O}_{6,18} & & & \\ \mathbf{O}_{3,6} & (M_2/M_{2'})\mathbf{I}_3 & \mathbf{O}_{3,3} & (M_3/M_{2'})\mathbf{I}_3 & \mathbf{O}_{3,3} & (M_4/M_{2'})\mathbf{I}_3 & \mathbf{O}_{3,3} \\ \mathbf{O}_{3,6} & (M_2/M_{2'})\mathbf{I}_3 & \mathbf{O}_{3,3} & (M_3/M_{2'})\mathbf{I}_3 & \mathbf{O}_{3,3} & (M_4/M_{2'})\mathbf{I}_3 & \mathbf{O}_{3,3} \end{bmatrix} \tag{54}$$

Evidently, both Eqs. (45) and (50) are in the form of Eq. (15), and $\mathbf{S}^{(2,1)}$ is a linear transformation on $\mathbb{R}^{24} \to \mathbb{R}^{12}$, implying state-space contraction. To apply Theorems 4.1 and 4.2 to the foregoing system, we need to derive estimates for $\|\mathbf{g}^{(1)}\|_{\mathbb{R}^{24}}$, $\|\mathbf{g}^{(2)}\|_{\mathbb{R}^{12}}$, and $\|\exp(\mathbf{A}_i t)\|^{(i)}$, $i = 1, 2$. First, the following estimates based on the ℓ_∞-norm for $\mathbb{R}^n$ can be readily derived using bounds (42–44), (52), and (53):

Lemma 5.1: Under conditions (42–44), (52), and (53), the nonlinear terms $\mathbf{g}^{(1)}$ and $\mathbf{g}^{(2)}$ in Eqs. (46) and (51) respectively satisfy the estimates

$$\|\mathbf{g}^{(1)}(t, \mathbf{x}^{(1)})\|_{\mathbb{R}^{24}} \leq \gamma_1(t)\|\mathbf{x}^{(1)}\|_{\mathbb{R}^{24}} \tag{55}$$

$$\|\mathbf{g}^{(2)}(t, \mathbf{x}^{(2)})\|_{\mathbb{R}^{12}} \leq \gamma_2(t)\|\mathbf{x}^{(2)}\|_{\mathbb{R}^{12}} \tag{56}$$

where

$$\gamma_1(t) = \frac{3\kappa_1(t)}{M_1} + \frac{\kappa_2(t)}{M_2} + \frac{\kappa_3(t)}{M_3} + \frac{3\kappa_4(t)}{M_4}$$
$$\gamma_2(t) = \frac{2\kappa_{1'}(t)}{M_{1'}} + \frac{\kappa_{2'}(t)}{M_{2'}} \tag{57}$$

To estimate $\|\exp(\mathbf{A}_1 t)\|_{\mathbb{R}^{24}}$ and $\|\exp(\mathbf{A}_2 t)\|_{\mathbb{R}^{12}}$, we note that the feedback controls (38) and (49) do not induce any loops in the signal flow between the spacecraft (see Fig. 1). Hence, the eigenvalues of $\mathbf{A}_1$ correspond to those of $\mathbf{Q}_i$, $i = 1, \ldots, 4$, and the eigenvalues of $\mathbf{A}_2$ correspond to those of $\mathbf{Q}_{1'}$ and $\mathbf{Q}_{2'}$. Because the feedback gain matrices $\mathbf{K}^{(P,i)}$ and $\mathbf{K}^{(R,i)}$ are diagonal, the eigenvalues of $\mathbf{Q}_i$ correspond to those of the 2×2 matrices:

$$\mathbf{Q}_{i,j} = \begin{bmatrix} 0 & 1 \\ -\omega_{ij}^2 & -2\zeta_{ij}\omega_{ij} \end{bmatrix}, \qquad \omega_{ij} = \sqrt{K_{jj}^{(P,i)}/M_i}$$
$$\zeta_{ij} = \frac{K_{jj}^{(R,i)}}{2\sqrt{M_i K_{jj}^{(P,i)}}}, \qquad j = 1, 2, 3, 4 \tag{58}$$

where $K_{jj}^{(P,i)}$ and $K_{jj}^{(R,i)}$ are the jth diagonal elements of $\mathbf{K}^{(P,i)}$ and $\mathbf{K}^{(R,i)}$, respectively. The eigenvalues of $\mathbf{Q}_{i,j}$ are given by $\lambda_{1,2} = (-\zeta_{ij} \pm \sqrt{\zeta_{ij}^2 - 1})\omega_{ij}$. To achieve nonoscillatory response, we set the feedback gains $K_{jj}^{(P,i)}$ and $K_{jj}^{(R,i)}$ such that the damping coefficient $\zeta_{ij} > 1$ or the eigenvalues of $\mathbf{Q}_{i,j}$ are negative real. Based on the ℓ_∞-norm for $\mathbb{R}^2$, we obtain

$$\|\exp(\mathbf{Q}_{i,j}t)\|_{\mathbb{R}^2} \leq \alpha_{ij}\exp(\beta_{ij}t), \qquad j = 1, 2, 3 \tag{59}$$

where

$$\alpha_{ij} = \frac{1 + \omega_{ij}\left(\zeta_{ij} + \sqrt{\zeta_{ij}^2 - 1}\right)}{2\omega_{ij}\sqrt{\zeta_{ij}^2 - 1}} \max\left\{2, \omega_{ij}\left(\zeta_{ij} + \sqrt{\zeta_{ij}^2 - 1} + \left|\zeta_{ij} - \sqrt{\zeta_{ij}^2 - 1}\right|\right)\right\}$$
$$\beta_{ij} = -\omega_{ij}\left(\zeta_{ij} - \sqrt{\zeta_{ij}^2 - 1}\right) \tag{60}$$

Using the foregoing estimates, we obtain the following bounds:

Lemma 5.2: Suppose that the feedback gains in $\mathbf{K}^{(P,i)}$ and $\mathbf{K}^{(R,i)}$, $i = 1, \ldots, 4; 1', 2'$ are chosen so that all eigenvalues of $\mathbf{Q}_{i,j}$, $i = 1, \ldots, 4, 1', 2'$; $j = 1, 2, 3$, are negative real, then

$$\|\exp(\mathbf{A}_1 t)\|_{\mathbb{R}^{24}} \leq \alpha_1\exp(\beta_1 t), \qquad \alpha_1 = \max_{i=1,\ldots,4;j=1,2,3}\{\alpha_{ij}\}$$
$$\beta_1 = \max_{i=1,\ldots,4;j=1,2,3}\{\beta_{ij}\}, \qquad t \in \bar{I}_1 \tag{61}$$

$$\|\exp(\mathbf{A}_2 t)\|_{\mathbb{R}^{12}} \leq \alpha_2\exp(\beta_2 t), \qquad \alpha_2 = \max_{i=1',2';j=1,2,3}\{\alpha_{ij}\}$$
$$\beta_2 = \max_{i=1',2';j=1,2,3}\{\beta_{ij}\}, \qquad t \in \bar{I}_2 \tag{62}$$

where α_{ij} and β_{ij} are given in Eq. (60).

From Eq. (54), we have $\|\mathbf{S}^{(2,1)}\| = 1$. Now, the following result follows directly from Lemma 5.1, Lemma 5.2, Theorem 4.1, and Theorem 4.2:

Preposition 5.1: The zero state of the switched semidynamical system equations (45) and (50) with feedback controls (38) and (49) and concatenation operator (54) is asymptotically stable in the sense of Lyapunov, if

$$-\alpha_2\bar{\gamma}_2 < \beta_2 < 0, \qquad \bar{\gamma}_2 = \lim_{t\to t_2}\frac{1}{(t - t_1)}\int_{t_1}^{t}\gamma_2(\tau)\,\mathrm{d}\tau \tag{63}$$

Moreover, if

$$\|\mathbf{x}^{(1)}(t_0)\|^{(1)} \leq \min\left\{\frac{\epsilon}{\sigma}, \frac{\mu}{\kappa}\right\} \tag{64}$$

where

$$\sigma = \alpha_1\alpha_2 \sup_{t\in I_2}\left\{\exp\left[\left(\beta_2 + \frac{\alpha_2}{t-t_1}\int_{t_1}^{t}\gamma_2(\tau)\,\mathrm{d}\tau\right)(t-t_1) + (\beta_1 + \alpha_1\bar{\gamma}_1)(t_1-t_0)\right]\right\} \tag{65}$$

$$\kappa = \alpha_1\alpha_2 \exp\left[\sum_{k=1}^{2}(\alpha_k+\beta_k)(t_k-t_{k-1})\right] \tag{66}$$

then the zero state is finite-time stable with specified bounds μ and ϵ with $\mu < \epsilon$.

The preceding result may be used to estimate the accuracy of formation alignment for feedback controls (38) and (49) with specified feedback gains and also the effect of persistent disturbances on formation stability.

Example: Let $M_1 = 40$ kg, $M_2 = M_3 = 10$ kg, $M_4 = 20$ kg. Thus, $M_1 = M_{1'} = 40$ kg, $M_{2'} = M_2 + M_3 + M_4 = 40$ kg. For the reference spacecraft 1 or 1′, we set $\omega_{1j} = 1$ rad/s, and $\zeta_{1j} = 1.1$, $j = 1, 2, 3$. Then, the corresponding feedback gains are $K_{jj}^{P,1} = 40$, $K_{jj}^{R,1} = 88$, $j = 1, 2, 3$. For spacecraft 2 and 3 before docking takes place, we set $\omega_{ij} = 2$ rad/s and $\zeta_{ij} = 1.1$, $i = 2, 3$; $j = 1, 2, 3$. The required feedback gains are $K_{jj}^{P,i} = 40$, $K_{jj}^{R,i} = 44$, $i = 2, 3$; $j = 1, 2, 3$. Finally, for spacecraft 4, setting $\omega_{ij} = 20$ rad/s and $\zeta_{ij} = 1.1$ lead to feedback gains $K_{jj}^{P,4} = 45$, $K_{jj}^{R,4} = 66$, $j = 1, 2, 3$. The feedback gains for spacecraft 1′ and 2′ after docking are identical to those for the reference spacecraft. Using the foregoing parameter values and Eq. (60), we obtain the following values for the parameters α_{ij}, β_{ij}, $i = 1, 2$ or $1', 2'$, $j = 1, 2, 3$:

$$\alpha_{1j} = 3.9805, \quad \alpha_{2j} = \alpha_{3j} = 9.8811, \quad \alpha_{4j} = 8.0111$$
$$\beta_{1j} = -0.6417, \quad \beta_{2j} = \beta_{3j} = -1.2835, \quad \beta_{4j} = -0.96261$$
$$\alpha_{1'j} = 3.9805, \quad \beta_{2'j} = -1.2835 \tag{67}$$

It follows from Eq. (60) that $\alpha_1 = \max\{\alpha_{ij}\} = 9.8811$, $\beta_1 = \max\{\beta_{ij}\} = -0.6417$, $\alpha_{2'} = 9.8811$, and $\beta_{2'} = -1.2835$. Thus, we can apply Preposition 5.1 to obtain explicit sufficient conditions for stability.

VI. Conclusions

In this work, we have obtained sufficient conditions for Lyapunov and finite-time stability with specified bounds for a special class of switched semidynamical systems with state-space dilation and contraction. Although we have only discussed the application of these results to a specific aerospace system, they are also applicable to other systems such as nanodynamical systems and reconfigurable microelectromechanical systems involving changes in the degrees of freedom [14]. Further studies in this area may consider the stability analysis and stabilizing feedback control design of other classes of switched semidynamical systems with state-space dilation and contraction.

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